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ON THE APPLICATION OF THE ENERGY METHOD TO STABILITY PROBLEMS

By Karl Marguerre

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ON THE APPLICATION OF THE ENERGY METHOD TO STABILITY PROBLEMS[1]

By Karl Marguerre

Since stability problems have come into the field of vision of engineers, energy methods have proved to be one of the most powerful aids in mastering them. For finding the especially interesting critical loads special procedures have evolved that depart somewhat from those customary in the usual elasticity theory. A clarification of the connections seemed desirable, especially with regard to the postcritical region, for the treatment of which these special methods are not suited as they are.

The present investigation[2] discusses this question -- complex (made important by shell construction in aircraft) especially in the classical example of the Euler strut, because in this case — since the basic features are not hidden by difficulties of a mathematical nature — the problem is especially clear.

The present treatment differs from that appearing in the Z.f.a.M.M. (1938) under the title "Über die Behandlung von Stabilitätsproblemen mit Hilfe der energetischen Methode" in that, in order to work out the basic ideas[3] still more clearly, it dispenses with the investigation of behavior at "large" deflections and of the elastic foundation; in its place the present version gives an elaboration of the 6th section and (in its 7th and 8th secs.) a new example that shows the applicability of the general criterion to a stability problem that differs from that of Euler in many respects.

SUMMARY

In the two examples of the Euler strut and the slightly curved beam under transverse load it was shown that the difference between the stability problems and the problems of linearized elasticity theory rests upon the fact that in the stability problems the expression for the energy of deformation contains terms of higher than the second order in the displacements. This idea makes it possible to establish the connection between the energy method in the special form most used for stability investigations and the principle of virtual displacements in its general elasticity -- theoretical version; besides, it permits the investigation of elastic behavior beyond the critical deflection.

INTRODUCTION

Kirchoff's uniqueness law states: An elastic body can assume one and only one equilibrium configuration under a given (sufficiently small) external loading. In the formulation from the energy point of view: the potential Π of the inner and outer forces has one and only one extremal

$$\delta\Pi = 0 \tag{1.1}$$

and the extremal is a minimum.[1]

The uniqueness law holds without restrictions in the realm of linearized theory of elasticity, that is, as long as the stresses σ, τ can be expressed linearly in terms of the strains γ_{ik} and the strains linearly in terms of the displacements u, v, w. Then the function Π is of, at most, the second degree in the displacements, and geometrical considerations show directly that a "parabola" of the second degree (positive definite quadratic form) can have one and only one minimum (or in mechanical terms, equilibrium position). The situation changes, however, when structures are considered the behavior of which can no longer be expressed with

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[1]For the derivation of the principle of virtual displacements (equation (1.1)) for elastic equilibrium, see, for example, reference 1, pp. 70 ff. A careful foundation of the general theory of the behavior at the stability limit appears in reference 2, p. 160. (For further literature, see reference 1, pp. 277 ff., or reference 2). The investigations in section Stable and Unstable Equilibrium in particular make use of the Trefftz point of view.

sufficient accuracy by the linearized strain-displacement equations. Such are, particulary, bodies for which one dimension is small compared to the others, structures in the form of shells, plates, or bars. For example, a rod can, without exceeding the proportional limit, undergo bending deflections several times greater than its thickness, and under these circumstances the quadratic part of the (transverse) displacements in the strain displacement equations is no longer small compared to the linear part. Then the energy of deformation of the potential II becomes of higher than the second degree in the displacements, and a parabola of higher order can naturally have several extremals (equilibrium positions).

The problem of the theory of stability is usually considered to be the determination of that external load under which several neighboring equilibrium configurations are possible. The reason for limiting investigation to this "critical point" lies in the fact that the differential equations describing the elastic behavior in the postcritical region are, in general, no longer linear and an analytical treatment would therefore be difficult; while at the critical point itself the problem can still be "linearized".[1]

This purely practical viewpoint has, however, led to a certain (as is shown, from unfounded standpoint) systematic separation of the stability problem from the other problems of the theory of elasticity, which finds its mathematical expression in a formulation of the principle of virtual displacements somewhat different from the usual one — has also for convenience led to the formulation of a special principle. (For example, see reference 3.)

The principle of virtual displacements states that during a virtual (that is, geometrically possible) displacement from an equilibrium position, the energy of deformation taken up by the elastic body is equal to the work done by the external forces. For the use of this principle in the theory of elasticity it is convenient to express this fact in the following way: An equilibrium state is distinguished by the fact that for every virtual displacement from that state the potential of the inner and outer forces

$$II = A_i + V$$

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[1] Knowledge of the postcritical region was until now of secondary practical interest, because buckled structural elements were considered unpermissible. It has been only in recent years that in the shell construction of aircraft critical loads have been permitted to be exceeded by large amounts unhesitatingly.

has a stationary value:

$$\delta \Pi = \delta(A_i + V) = 0 \qquad (1.2)$$

Therein the potential A_i of the inner forces is given by the energy of deformation (inner work), and the potential V of the external forces by the negative product of the external forces considered constant and the displacements of their points of application. In the region of applicability of the proportionality law numerically $V = -2A_a$, where A_a is the work done by the external forces as they increase from zero to their final values in passing through only equilibrium states. The principle (1.2) can therefore be written conveniently also in the form. (For example, see reference 3.)

$$\delta(A_i - 2A_a) = 0 \qquad (1.3)$$

As against this there is often used as a "minimal principle" in stability theory the condition

$$\delta(A_i - A_a) = 0 \qquad (1.4)$$

The author intends to show in the present paper (in the classical example of the strut) that also stability investigations are best handled in connection with the single main equation (1.2), wherein terms must be retained of higher order in the deformations logically only in the expression for the energy of deformation. This procedure is essential from the practical standpoint, if the relationships are to be investigated in the postcritical region, and desirable from the systematic standpoint, because it becomes clear in this manner that no additional principles are required. In particular, this consideration will clear up the apparent contradiction between equations (1.3) and (1.4).

The calculation itself is carried out in the following manner: First, the expression for the energy of deformation A_i is set up, then the differential equations for the two components of displacement are derived from the condition

$$\delta \Pi = 0$$

and the question of the stability of the equilibrium position is answered
by the restricted condition

$$\Pi = \text{minimum} \qquad (1.5)$$

Then the same problem is treated with the help of a Ritz procedure; in
this way the result of the stability consideration is brought out in an
especially elegant manner. After a thorough discussion of the usual sta-
bility theory, it is shown in conclusion how the same considerations can
serve for the treatment of the snap-action problem of a slightly curved
beam.

ENERGY OF DEFORMATION

If the customary assumptions of the beam theory are retained — that
for small deflections of the beam the work of stretching and work of
bending are independent of each other and that the part of the work re-
sulting from the shear forces is small compared with the other two parts —
then the energy of deformation can easily be given.

As a result of the assumption of small displacements — without at first
saying anything about the sizes of the displacements u and w (fig. 1)
relative to each other — the strains u_x and w_x (or w_x^2) may be neglected
in comparison with unity; that is, in a development of both quantities in
powers of u_x and w_x only the lowest power need be retained. If, there-
fore, the square of a line element of the beam centerline before deforma-
tion is

$$dx^2$$

and after deformation

$$\left[\left(1 + u_x \right)^2 + w_x^2 \right] dx^2 \qquad (2.1)$$

(the subscripts on u and w indicate derivatives with respect to x),
then the strain of the beam centerline (reference 1, p. 57) is

$$\overline{\epsilon}_x = \sqrt{\left(1 + u_x \right)^2 + w_x^2} - 1 \approx u_x + \frac{w_x^2}{2} \qquad (2.2)$$

From Hooke's law the corresponding stress is

$$\overline{\sigma}_x = E\left(u_x + \frac{w_x^2}{2} \right) \tag{2.3}$$

and therefore the energy of stretching is

$$\overline{A} = \frac{EF}{2} \int_0^l \left(u_x + \frac{w_x^2}{2} \right)^2 dx \tag{2.4}$$

The incremental strain $\widetilde{\epsilon}_x$ due to bending is, according to the assumption that normals are preserved,

$$\widetilde{\epsilon}_x = -zw_{xx} \tag{2.5}$$

the bending energy is therefore given by

$$\widetilde{A} = \frac{E}{2} \iiint \left(zw_{xx} \right)^2 dx\, dy\, dz = \frac{EJ}{2} \int_0^l w_{xx}^2 dx \tag{2.6}$$

If $\epsilon l = u(0) - u(l) = -u(l)$ is the distance of approach of the ends, $P = pF = -\overline{\sigma}_x F$ the compressive force, then $-(pF)(\epsilon l) = V$ is the potential of the external forces; the total potential Π (measured from the stress-free state as the zero position just as the displacements u, w, ϵ!) is therefore

$$\Pi = \frac{EF}{2} \int_0^l \left(u_x + \frac{w_x^2}{2} \right)^2 dx + \frac{EJ}{2} \int_0^l w_{xx}^2\, dx - pF\epsilon l \tag{2.7}$$

The potential per unit length — after division by EF — can be written

$$\widehat{\Pi} = \frac{\Pi}{EFl} = \frac{1}{2l} \int_0^l \left(u_x + \frac{w_x^2}{2} \right)^2 dx + \frac{i^2}{2l} \int_0^l w_{xx}^2\, dx - \frac{p}{E} \epsilon \tag{2.8}$$

From equation (2.8) and the condition (1.5)

$$\hat{\Pi} = \text{minimum} \qquad (2.9)$$

there is obtained all information about the behavior of the strut at and beyond the stability limit.

THE DIFFERENTIAL EQUATIONS FOR THE DISPLACEMENTS u, w

Consider a rod the left end of which is $(x = 0)$ is supported and the right end gives $(x = l)$ is freely movable in a horizontal direction under a centrally placed compressive force P. (See fig. 1.) As given (that is, as the independent variable of the problem) take either the horizontal displacement of the right-hand end | or the compressive stress

$$u(l) = - \epsilon l \qquad\qquad\qquad p = \frac{P}{F}$$

According to the principle of virtual displacements, the displacements are to be varied under a constant load in a manner compatible with the geometrical conditions. If at a boundary point the displacement (in the present problem, for example, ϵ) is prescribed, the point is held fixed during the variation, so that the work of the (unknown!) external forces does not remain in the calculation; if on the other hand, the force is given, then the (not fixed geometrically) end point is varied, and the work of the external load (in the present problem Pδu) enters into the calculation.

If in equation (2.8) the displacement u is varied (that is, if the elements of the rod are given a virtual displacement in the axial direction, while the boundary point or correspondingly the load is held fixed) there follows from the rules of the calculus of variations

$$\frac{\partial}{\partial x} \left(u_x + \frac{wx^2}{2} \right) = 0$$

together with the boundary conditions   (3.1)

$$u(0) = 0 \qquad\qquad\qquad u(0) = 0$$

$$u(l) = - \epsilon l \qquad\qquad \left[u_x + \frac{wx^2}{2} + \frac{p}{E} \right]_{x=l} = 0$$

by varying w there is obtained (independently of whether ϵ or p is considered as given):

$$-w_x \frac{\partial}{\partial x}\left(u_x + \frac{w_x^2}{2}\right) - \left(u_x + \frac{w_x^2}{2}\right)w_{xx} + 1^2 w_{xxxx} = 0$$

with the boundary conditions   (3.2)

$$w(0) = w(l) = w_{xx}(0) = w_{xx}(l) = 0$$

The exact integration of the system (3.1), (3.2) of nonlinear (!) simultaneous equations offers no difficulty here.

From (3.1) there follows

$$u_x + \frac{w_x^2}{2} = \text{constant} = -\epsilon_0$$

(3.3)

$$u = u(0) - \epsilon_0 x - \int_0^x \frac{w_x^2}{2}\, dx$$

and with the use of (3.3), (3.2) becomes

$$w_{xxxx} + \frac{\epsilon_0}{1^2} w_{xx} = 0$$ (3.4)

The solution of this linear equation with constant coefficients is

$$w = f \sin kx + g \cos kx + f_1 x + g_1$$

where k is the positive root of the quadratic equation

$$k^2 - \frac{\epsilon_0}{1^2} = 0$$

For the determination of the six constants of integration f, g, f_1, g_1, $u(0)$, ϵ_0 there are the six boundary conditions (3.1) and (3.2):

$$g + g_1 = 0 \qquad\qquad k^2 g = 0$$

$$f \sin kl + g \cos kl + f_1 l + g_1 = 0$$

$$fk^2 \sin kl + gk^2 \cos kl = 0$$

$$u(0) = 0$$

and

$$- \epsilon l = u(0) - \epsilon_0 l - \int_0^l \frac{w_x^2}{2} \, dx \qquad\Bigg|\qquad - \epsilon_0 + \frac{p}{E} = 0 \qquad (3.5)$$

It is found that

$$f_1 = g_1 = g = u(0) = 0$$

and either

$$f = 0, \quad \text{that is,} \quad w = 0$$

$$\epsilon_0 = \epsilon \qquad\Bigg|\qquad \epsilon_0 = \frac{p}{E}$$

$$u = - \epsilon x \qquad\Bigg|\qquad u = - \frac{p}{E} x \qquad\qquad (3.6)$$

or

$$f \neq 0, \quad w = f \sin kx, \quad k = \frac{\pi}{l}$$

$$\epsilon_0 = \frac{i^2 \pi^2}{l^2} \equiv \epsilon^* \qquad\qquad (3.7)$$

$$u = - \epsilon^* x + \frac{\pi}{8} \frac{f^2}{l} \sin \frac{2\pi x}{l}$$

There are evidently two kinds of equilibrium positions: the straight $(f = 0)$ and the bent $(f \neq 0)$. The straight position is specified uniquely by either p or ϵ, the bent by ϵ — for the amplitude f of the deflection can be determined uniquely from the left one of equations (3.5)

$$\frac{\pi^2 f^2}{4l^2} = \epsilon - \epsilon_0 = \epsilon - \epsilon^* \qquad\qquad (3.8)$$

not, however, by p. From the right one of equations (3.5) there follows
rather that for $f \neq 0$ a completely determined "critical" value

$$p = E\epsilon^* \equiv p^* \tag{3.9}$$

cannot be exceeded. Therefore p is unsuited for an independent
parameter (the situation is different in the case of the corresponding
plate problem) (reference 4, p. 124). From equation (3.8) it can be seen
that f assumes real values only for $\epsilon > \epsilon^*$.

In figures 2 and 3 the quantities f and p are plotted against ϵ.
The solid lines are for the solution (3.7), the dotted lines for equation
(3.6).

The result (3.9) — that the load for the buckled strut does not in-
crease beyond p* even when the critical end shortening has been con-
siderably exceeded — is a result of the limitation to "small" deflections.[1]
For the present problem this limitation is not important because here it
was only a question of seeing that as a result of the appearance of higher
powers of w in equation (2.8) the elastic rod can assume several equi-
librium positions — especially that the existence of a real multiplicity
is bound up with the exceeding of a certain "critical" strain $\epsilon = \epsilon^*$.

STABLE AND UNSTABLE EQUILIBRIUM

In the "ordinary" theory of elasticity it is necessary to consider
only the condition

$$\delta\Pi = 0, \quad \text{that is,} \quad \Pi = \text{extremum} \tag{4.0}$$

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[1]Reference 5, pp. 70 ff. Also the theory of the so-called exact
differential equation of strut buckling

$$EJ/\rho + Pw = 0$$

(reference 1, p. 280) shows that at large deflections (because of the in-
creased demand on bending energy) there is a very small increase in load.
From the energy standpoint, to be sure, this "exact" equation is not im-
portant, for if w_x^2 is taken as not small compared to 1 in the bend-
ing term (that is, the curvature $1/\rho$ is used in place of w_{xx}) it is
necessary to proceed in a corresponding manner with the stretching term
(equation (2.2)) — unless it is assumed that an incompressible strut exists
from the start, as is done in the theory of the Euler elastica.

in the determination of equilibrium states, for the supplementary condition $\delta^2\Pi > 0$ (mechanically: the stability of the equilibrium position) is assured there because of the linearization (reference 1, pp. 71--72); in the present problem the minimal condition must be set up explicitly, since only through

$$\delta\Pi = 0, \quad \delta^2\Pi > 0, \quad \text{that is,} \quad \Pi = \text{minimum} \qquad (4.1)$$

can the stable equilibrium positions be distinguished from other possible (the unstable) positions. The concept of stability is made precise here by the following convention.[1]

1. An equilibrium state is called stable if for every neighboring state the potential energy has a larger value.

2. An equilibrium state is called labile (unstable) if there is at least one neighboring state for which the potential energy is smaller.

3. A stability limit (that is, a neutral equilibrium state) is spoken of when there is at least one neighboring equilibrium state the potential energy of which is equal to but none having potential energy less than that of the given equilibrium state.

Return to equations (2.8) or (2.9):

$$\hat{\Pi} = \frac{1}{2l}\int_0^l \left(u_x + \frac{w_x^2}{2} \right)^2 \, dx + \frac{i^2}{2l}\int_0^l w_{xx}^2 \, dx - \frac{p}{E}\,\epsilon \overset{!}{=} \text{minimum}$$

and perform a variation; that is, replace u by $u + \delta u$ and w by $w + \delta w$; there results, after arranging in powers of δu, δw and stopping after terms of the second order,

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[1] The following definition was given, in substance by E. Trefftz incidentally to his DVL lecture.

$$\Delta\hat{\Pi} = \hat{\Pi}(u + \delta u,\ w + \delta w) - \hat{\Pi}(u,w) = \delta\hat{\Pi} + \frac{1}{2}\delta^2\hat{\Pi} + \ldots$$

$$= -\frac{1}{l}\frac{p}{E}\delta u\ \Big|_{\,x=l} + \frac{1}{l}\int_0^l \left(u_x + \frac{w_x^2}{2}\right)\delta u_x\,dx$$

$$+ \frac{1}{l}\int_0^l \left[\left(u_x + \frac{w_x^2}{2}\right)w_x\,\delta w_x + l^2 w_{xx}\,\delta w_{xx}\right]dx$$

$$+ \frac{1}{2l}\int_0^l \left(\delta u_x\right)^2 dx + \frac{1}{l}\int_0^l w_x\,\delta u_x\,\delta w_x\,dx$$

$$+ \frac{1}{2l}\int_0^l \left[\left(u_x + \frac{w_x^2}{2} + w_x^2\right)\left(\delta w_x\right)^2 + l^2\left(\delta w_{xx}\right)^2\right]dx$$

The condition that the terms of first order shall vanish leads to equations (3.1) and (3.2); the question of stability is answered by the terms of second order. By inserting for u, w the values obtained from $\delta\Pi = 0$ there results for (3.6) (straight position)

$$\left(\delta^2\hat{\Pi}\right)_1 \equiv \left(\delta^2\hat{A}_1\right)_1 = \frac{1}{l}\int_0^l \left[-\epsilon\left(\delta w_x\right)^2 + l^2\left(\delta w_{xx}\right)^2\right]dx + \frac{1}{l}\int_0^l\left(\delta u_x\right)^2 dx \quad (4.2_1)$$

for (3.7) (bent position)

$$\left(\delta^2\hat{\Pi}\right)_2 \equiv \left(\delta^2\hat{A}_1\right)_2 = \frac{1}{l}\int_0^l \left[\left(-\epsilon* + \frac{\pi^2}{l^2}f^2\cos^2\frac{\pi x}{l}\right)\left(\delta w_x\right)^2 + l^2\left(\delta w_{xx}\right)^2\right]dx$$

$$+ \frac{2\pi f}{l^2}\int_0^l \cos\frac{\pi x}{l}\,\delta w_x\,\delta u_x\,dx + \frac{1}{l}\int_0^l\left(\delta u_x\right)^2 dx \quad (4.2_2)$$

First the stability limit will be determined. According to the definition given above there must be at the stability limit a state for which the second variation vanishes but none for which it becomes negative: The value zero is therefore the smallest value that $\delta^2\Pi$ can assume at the limiting point. If, therefore, $\delta^2\Pi$ has certain continuity properties (the existence of which is obvious on mechanical grounds), then

the "characteristic" value $\delta^2 \Pi = 0$ is at the same time an analytical minimum, compared with neighboring values, and the associated ("characteristic") displacement system δu, δw is determined from the condition

$$\delta\left(\delta^2 \hat{\Pi}\right) = 0 \qquad (4.3)$$

The straight position (see equation (4.2_1)) thus reaches the stability limit when δu, δw satisfy the two differential equations

$$(\delta u)_{xx} = 0$$

$$(\delta w)_{xxxx} + \frac{\epsilon}{i^2}(\delta w)_{xx} = 0 \qquad (4.4)$$

with the end conditions

$$\delta w(0) = \delta w(l) = \delta w_{xx}(0) = \delta w_{xx}(l) = \delta u(0) = 0$$

and

$$\delta u(l) = 0 \qquad \Big| \qquad \delta u_x(l) = 0 \qquad (4.5)$$

The solution of this eigenvalue problem reads in both cases

$$\delta u = 0, \quad \delta w = \delta f \sin \frac{\pi x}{l} \qquad (4.6)$$

The amplitude $\delta f \neq 0$ remains undetermined and from the second of equations (4.4) there is obtained for the critical value of

$$\epsilon_{crit} = \frac{\pi^2 i^2}{l^2}$$

that is exactly the expression $\epsilon_{crit} = \epsilon^*$ by which the branch point of the equilibrium was characterized; stability limit and branch point coincide.

The investigation proves to be somewhat more difficult for the bent position. The two minimal conditions read

$$\frac{\partial}{\partial x}\left(\delta u_x + \frac{\pi f}{l} \cos \frac{\pi x}{l} \delta w_x \right) = 0$$

$$(4.7)$$

$$\frac{\partial}{\partial x}\left[l^2 \delta w_{xxx} + \left(\epsilon^* - \frac{\pi^2 f^2}{l^2} \cos^2 \frac{\pi x}{l} \right) \delta w_x - \frac{\pi f}{l} \cos \frac{\pi x}{l} \delta u_x \right] = 0$$

for the boundary condition there is retained

$$\delta w(0) = \delta w(l) = \delta w_{xx}(0) = \delta w_{xx}(l) = \delta u(0) = 0$$

and, depending on whether or not the right-hand boundary point is pre-scribed or not,

$$\delta u(l) = 0 \qquad \Big| \qquad \left(\delta u_x + \frac{\pi f}{l} \cos \frac{\pi x}{l} \delta w_x \right)_{x=l} = 0 \qquad (4.7')$$

It is recognized immediately that for $f = 0$, hence at the beginning of buckling, the bar is in neutral equilibrium, for all conditions are satis-fied by the solution (4.6). This result is trivial. It is not so directly obvious, however, that also for $f \neq 0$ there is a variation that makes $\delta^2 \Pi$ a minimum; the homogeneous system of equations with homogeneous boundary conditions permits of a non-vanishing solution also for $f \neq 0$. In fact if the variation is so performed that the second boundary con-dition of (4.7') is satisfied there follows from the first of equations (4.7)

$$\delta u_x = - \frac{\pi f}{l} \cos \frac{\pi x}{l} \delta w_x \qquad (4.8)$$

If this is put in the second of equations (4.7) the latter reduces to

$$l^2 \delta w_{xxxx} + \epsilon^* \delta w_{xx} \equiv l^2 \left(\delta w_{xxxx} + \frac{\pi^2}{l^2} \delta w_{xx} \right) = 0$$

and this equation (together with its boundary conditions) is satisfied by $\delta w = \delta f \sin \frac{\pi x}{l}$. For δu there is obtained from (4.8)

$$\delta u = - \frac{\pi^2 f \delta f}{2 l^2} \left(x + \frac{l}{2\pi} \sin \frac{2\pi x}{l} \right).$$

in which δf again represents the (not determinable by a system of homogeneous equations) arbitrary factor.

The question of the stability of the equilibrium positions below and above the limit can now be answered.

1. Since the straight position (see equation (4.2_1)) is stable for very small ϵ, from considerations of continuity it is so for $\epsilon \leq \epsilon^*$.

2. For $\epsilon > \epsilon^*$ the straight position represents an unstable equilibrium state, for a variation δw, namely,

$$\delta w = \delta f \sin \frac{\pi x}{l}$$

can be given for which $\delta^2 \Pi < 0$.

3. For $\epsilon > \epsilon^*$ the bent rod is against the variation

$$\delta w = \delta f \sin \frac{\pi x}{l}$$
$$\delta u = - \frac{\pi^2 f \delta f}{2 l^2} \left(x + \frac{l}{2\pi} \sin \frac{2\pi x}{l} \right)$$

(4.9)

in neutral equilibrium.[1] Since the variation (4.9) (and only this) makes $\delta^2 \Pi$ a minimum (of value zero) every other variation gives it a positive value. If in some way the special variation (4.9) is prevented, then the bent equilibrium position is stable. This holds especially in the important case where not the force but the displacement of the end point is prescribed; for the displacement system (4.9) is in fact excluded by the boundary condition $\delta u(l) = 0$.

At this point a result discussed later in section Connection between the Ordinary Investigation of Stability and the Procedure Presented Here, should be emphasized. The behavior of the rod beyond the stability limit is different depending upon whether the load or the displacement is

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[1] This result is naturally connected with the assumption that the strut adheres strictly to the law of deformation established by the expression (2.7).

considered as the prescribed quantity. More noteworthily, however, the
behavior at the stability limit is not affected by this difference. For,
although there are the two different boundary conditions (4.5)

$$\delta u(l) = 0$$

and

$$\delta u_x(l) = 0$$

they both (together with the differential equation $\delta u_{xx} = 0$) lead to
the same result

$$\delta u \equiv 0$$

That is, it makes no difference whether a motion of the end points in the x-
direction is "permitted" or not: during the buckling they do not move.
Thus the result is arrived at that the two mechanically entirely different
problems: buckling under constant load and buckling under constant end
shortening, lead to exactly the same critical state δu, δw, ϵ_{crit}, p_{crit}.[1]

INTERPRETATION OF THE RESULTS WITH AID OF THE RITZ METHOD

The results of sections The Differential Equations for the Displace-
ments u, w and Stable and Unstable Equilibrium can be illustrated very
elegantly if the variation problem is turned into an ordinary minimum
problem by the use of the Ritz method. In the case of the Ritz method
to be sure nothing certain can be said about the question of stability,
since from the start only quite definite displacements are considered and
therefore nothing general can be concluded about the sign of the second varia-
tion; nevertheless, with a judiciously chosen deflection system the
question can be answered with great probability or the answer made very
plausible. In the present, especially simple case the earlier results
are found again exactly.

As a set of displacements satisfying all boundary conditions are
chosen, the solutions of the differential equations (3.2) and (3.1)

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[1]This is a peculiarity of the problem. In general, the critical load
depends upon the boundary conditions — whether are prescribed forces or
displacements, the system is supported, or guided, or built in, etc.; for
the "minimum" — variation, from which the critical load follows, differs
according to the boundary conditions prescribed by the data of the problem.

$$w = f \sin \frac{\pi x}{l}, \quad u_x + \frac{w_x^2}{2} = -\epsilon + \frac{\pi^2 f^2}{4 l^2} \tag{5.1}$$

Then the minimal condition

$$\hat{\Pi} = \frac{1}{2l} \int_0^l \left(u_x + \frac{w_x^2}{2} \right)^2 dx + \frac{i^2}{2l} \int_0^l w_{xx}^2 \, dx - \frac{p}{E} \epsilon$$

$$\tag{5.2}$$

$$= \frac{\pi^4 f^4}{32 l^4} - \frac{\pi^2 f^2}{4 l^2} (\epsilon - \epsilon^*) + \frac{\epsilon^2}{2} - \frac{p}{E} \epsilon \stackrel{!}{=} \text{minimum}$$

furnishes an equation for the "free value" f as a function of ϵ

$$\frac{\partial \hat{\Pi}}{\partial f} \equiv \frac{f \pi^2}{2 l^2} \left(\epsilon^* - \epsilon + \frac{\pi^2 f^2}{4 l^2} \right) = 0 \tag{5.3}$$

The relationship between load and end displacement is obtained by means of the stress-strain equation (2.3) from the second of equations (5.1); it becomes

$$\frac{p}{E} = \epsilon - \frac{\pi^2 f^2}{4 l^2} \tag{5.4}$$

There are again the two possible solutions

$$f = 0, \quad p = E\epsilon \tag{5.5}$$

and

$$f \neq 0, \quad \text{hence} \quad \frac{\pi^2 f^2}{4 l^2} = \epsilon - \epsilon^*, \quad p = E\epsilon^* \tag{5.6}$$

The question of stability is answered (with the above-mentioned limitation) by the second variation. As hitherto, two cases are distinguished:

1. If the force is prescribed, ϵ therefore left open, there are two displacements to be varied, and the sign of the expression

$$\delta^2 \hat{\Pi} = \frac{\partial^2 \hat{\Pi}}{\partial f^2} (\delta f)^2 + 2 \frac{\partial^2 \hat{\Pi}}{\partial f \partial \epsilon} \delta f \delta \epsilon + \frac{\partial^2 \hat{\Pi}}{\partial \epsilon^2} (\delta \epsilon)^2 \qquad (5.7)$$

must be investigated. This expression is a quadratic form in δf, $\delta \epsilon$ with the coefficients

$$\frac{\partial^2 \hat{\Pi}}{\partial f^2} = \frac{\pi^2}{2l^2} \left(\epsilon^* - \epsilon + 3 \frac{\pi^2 f^2}{4l^2} \right)$$

$$\frac{\partial^2 \hat{\Pi}}{\partial f \partial \epsilon} = - \frac{f \pi^2}{2l^2}$$

$$\frac{\partial^2 \hat{\Pi}}{\partial \epsilon^2} = 1$$

Since $\dfrac{\partial^2 \hat{\Pi}}{\partial \epsilon^2} > 0$, this expression is positive definite (that is, never negative) as long as the discriminant

$$\Delta = \frac{\partial^2 \hat{\Pi}}{\partial f^2} \frac{\partial^2 \hat{\Pi}}{\partial \epsilon^2} - \left(\frac{\partial^2 \hat{\Pi}}{\partial f \partial \epsilon} \right)^2 = \frac{\pi^2}{2l^2} \left(\epsilon^* - \epsilon + \frac{\pi^2 f^2}{4l^2} \right) \qquad (5.8)$$

is greater than zero. This is certainly the case below the critical point $(\epsilon < \epsilon^*)$; here the system is therefore stable. On the other hand, $\Delta < 0$ for $f = 0$ and $\epsilon > \epsilon^*$; that is, the straight position is unstable above the critical point. Finally, the bent position is neutral, because by (5.6) the discriminant vanishes for $f \neq 0$. This result agrees with that of the previous section and is an expression of the fact that a buckled strut in the elastic range can be bent arbitrarily farther without increasing the load.

2. If the end displacement ϵ is prescribed, only the quantity f need be varied and it is found that:

$$\delta^2 \hat{\Pi} = \frac{\partial^2 \hat{\Pi}}{\partial f^2} (\delta f)^2 = \frac{\pi^2}{2l^2} \left(\epsilon^* - \epsilon + 3 \frac{\pi^2 f^2}{4l^2} \right) (\delta f)^2 \qquad (5.9)$$

From this relation it follows immediately that

$$\text{for } \epsilon < \epsilon^* \qquad \delta^2 \hat{\Pi} > 0$$

$$\text{for } \epsilon > \epsilon^* \text{ and } f \neq 0 \quad \delta^2 \hat{\Pi} > 0$$

$$\text{for } \epsilon > \epsilon^* \text{ and } f = 0 \quad \delta^2 \hat{\Pi} < 0$$

that is, the straight position is stable for $\epsilon < \epsilon^*$, unstable for $\epsilon > \epsilon^*$, the bent position, as soon as it is mechanically possible (hence for $\epsilon > \epsilon^*$), always stable. Figure 4 shows the energy relationships in this second case (prescribed displacement $\epsilon = \alpha\epsilon^*$ of the right-hand end of the rod). Plotted as $\hat{A}_1$

or

$$\frac{\hat{A}_1}{\epsilon^{*2}} = \frac{1}{2}\xi^4 - \xi^2(\alpha - 1) + \frac{\alpha^2}{2}$$

as a function of $f\left(\text{or } \xi = \dfrac{f}{2i}\right)$ with $\epsilon\left(\text{or } \alpha = \dfrac{\epsilon}{\epsilon^*}\right)$ as a parameter.

(The potential of the external forces is not included, because it is not affected by the minimal condition with respect to f.) It is seen that the straight position ($f = 0$) is an equilibrium position under all circumstances, for all curves start with a horizontal tangent. For $\alpha < 1$ associated with $f = 0$ is a minimum, for $\alpha > 1$ a maximum; the curves for $\alpha > 1$ have further to the right also a minimum, whereby the bent position $f \neq 0$ is characterized as a (stable) equilibrium position. This figure shows especially well the "type" change of the curves in the transition from the sub-critical to the supercritical region: the coinciding of maximum and minimum for $\alpha = 1$. It is also clear here that, although at the moment of transition the displacements are small, the behavior of the body at the stability limit is nevertheless determined by the "possibility" of greater deflections, expressed mathematically, by the existence of the f-terms of higher order in the expression for $\hat{A}_1$.

CONNECTION BETWEEN THE ORDINARY INVESTIGATION OF STABILITY

AND THE PROCEDURE PRESENTED HERE

The ordinary stability theory is limited to an investigation of the
critical point. It was seen that the critical point is characterized by
two energy conditions. The condition

$$\delta\Pi = 0$$

for any variation $\delta u, \delta v, \delta w$ (6.1)

characterizes it as an equilibrium position in general; the condition

$$\delta^2\Pi = 0$$

for a characteristic variation $\delta u, \delta v, \delta w$ (6.2)

as the critical one. Or, the critical point is distinguished by the fact
that there a variation of the state of deformation can be made for which
the potential Π remains unchanged to terms of the second order. Now
in practical buckling problems it is usually a question of the transition
from a very simple (often independent of the coordinates) initial state
of stress to a comparatively very complicated one. It is therefore custom-
ary to specify the initial state of stress directly without recourse to the
definition in terms of energy (6.1) and to proceed with the variation im-
mediately in regard to the determination of the second state. There then
remains as the single important condition the statement (6.1), which can
be expressed in the form of a method-of-procedure as follows, for example:
Consider a system of infinitesimal distortions superimposed upon the
critical state of deformation, collect the parts of the potential energy
Π quadratic in the added displacements and set the sum equal to zero.
That such a procedure is at all possible rests upon the fact that as a
result of the "large" initial stresses two types of quadratic term arise:
an (always positive) part that represents the work done by the stresses
caused by the added displacements, and a second part that comes from the
work done by the stresses already present upon the quadratic part of the
added displacements. (See reference 2.)[1]

The fact that this statement-of-procedure concerning the vanishing
of the quadratic members is nothing more than the extended principle of
virtual displacements ("extended" in the sense of the statement about
$\delta^2\Pi$) does not come out clearly in the applications mostly for three
reasons.

[1]In equation (4.2_1), for example, the last two terms represent the
first type and the first term, the second type.

1. Since a confusion with the very simple initial state is in general not to be feared, it is possible to dispense with the designation δu, δv, δw and write more briefly u, v, w for the added displacements. This manner of writing does not express the fact that the added displacements are to be not only small in the sense of the general hypotheses of the theory of elasticity but also infinitesimal in the sense of the calculus of variations.

2. In close connection with the above, in considering the energy it is customary to start not with the total potential Π but directly with the energy changes (appearing as the result of u, v, w) and to designate these changes by[1] A, V instead of by δA, δV; the (extended) principle of virtual displacements becomes in this manner of writing

$$A_i + V = 0, \quad \text{or even} \quad A_i = A_a \qquad (6.3)$$

since the potential difference $- V$ also represents the work of the external forces on the infinitesimal[2] displacements u, v, w. Equations (6.3) can be put into words as follows: For the virtual displacement u, v, w, through which the original equilibrium configuration goes over into the neighboring ("buckled") configuration at the critical point the internal energy A_i taken up by the system is equal to the work done by the external forces A_a taking into account the terms linear and quadratic in u, v, w. By this formulation the two conditions (6.1) and (6.2) are combined into one; a procedure in which there is the danger of losing sight of the difference between the (holding for any equilibrium position) principle (6.1) and the (characterizing the critical position) extension (6.2).

3. As the proper equation for the determination of the critical system of virtual displacements u, v, w there follows (see sec. Stable and Unstable Equilibrium) from (6.2) and the added requirement

$$\delta^2 \Pi > 0 \quad \text{for all other} \quad \delta u, \delta v, \delta w$$

the condition

$$\delta(\delta^2 \Pi) = 0 \qquad (6.4)$$

---

[1]To distinguish them from those used earlier, the quantities usually written A_i, A_a, V are designated by $\underline{A}i$, $\underline{A}_a$, $\underline{V}$.

[2]The second form of the law (6.3) therefore does not represent the special energy law $A_i = A_a$, by which is expressed the fact that for conservative systems the external work introduced by the transition from the initial to the (not neighboring!) final state is stored up as elastic energy in the body.

If it is agreed to consider in $\underline{A}_i$, $\underline{A}_a$, $\underline{V}$ only the (alone essential for the critical behavior) quadratic terms, the condition (6.4) is written in the form

$$\delta(\underline{A}_i + \underline{V}) = 0$$

$$\delta(\underline{A}_i - \underline{A}_a) = 0 \tag{6.4'}$$

This form, which is only a natural consequence of the original agreement to write u, v, w instead of δu, δv, δw, makes it quite clear to what extent the simplified manner of writing can lead to conceptual errors. For the statement (6.4'), aside from the deceptive formal agreement, has nothing to do with the principle of virtual displacements (1.2) or (1.3): The principle (1.2), in content the same as the energy law (see the Introduction), answers the question of the equilibrium positions under given loads (or edge displacements), and equation (1.3) is a special form of the same principle possible only in the realm of linearized elasticity theory besides being very inexpedient[1]; equation (6.4'), on the other hand, in content the same as the minimal condition (6.4) concerning the behavior of the quadratic terms at the stability limit, gives the second equilibrium position possible at the branch point and the sought-for value of the load at which the equilibrium begins to be many-valued.

The difficulties so far discussed were difficulties in interpretation arising from the symbolism of writing. There is another, more factual circumstance that makes the question complex especially difficult to see through. It was seen in section, Stable and Unstable Equilibrium that for the rod there were two independent equations (4.4), with the likewise independent boundary conditions (4.5), for the two added displacements δu, δw (which here would have been written u, w). From them it was concluded that u vanished identically. This result — and correspondingly $u \equiv 0$, $v \equiv 0$ in the case of plates — makes possible, when (as is tacitly done in the stability theory of a bar) it is presupposed as known, a treatment of the problems of bar and plate stability deviating from the general methods of stability theory depicted above. Since, however, bars and plates are the most well-known problems, being analytically the most tractable, frequently ideas that were developed there are erroneously brought into

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[1]So, for instance, for the compressed strut below the critical point twice the external work can be written in three forms: $E\epsilon^2$, $p\epsilon$, p^2/E — which is to be varied (with respect to ϵ!)? The second form is meant, but as a result of writing $2A_a$ in the place of $-V$ that is no longer uniquely discernible.

more general stability problems. It is therefore necessary to examine more thoroughly the various — special — interpretations that can be given to the occurrences at the stability limit in the case of bars and plates.

First of all outline the method by which it is necessary to proceed according to the directions formulated at the beginning of this section. If it is assumed a virtual displacement u , w at the critical point, then, as can easily be seen[1] the strain of a fiber to terms of the second order is given by

$$\epsilon_x = u_x + \frac{w_x^2}{2} - zw_{xx} = \bar{\epsilon}_x - zw_{xx}$$

Therefore the terms of second order are: in the work done by the added stress

$$\iiint \frac{E}{2} \left(u_x - zw_{xx} \right)^2 dx \, dy \, dz$$

in the work done by the already present (critical) compressive stress p

$$-\iiint p \frac{w_x^2}{2} dx \, dy \, dz$$

(The external force does work — $Pu(l)$; this makes no contribution of the second order.) After integration over y and z there results

$$A_1 = \frac{EF}{2} \left(\int_0^l u_x^2 \, dx + i^2 \int_0^l w_{xx}^2 \, dx - \frac{p}{E} \int_0^l w_x^2 \, dx \right) \quad A_2 = 0 \qquad (6.5)$$

and the conditions (6.3) and (6.4') become

$$\int_0^l u_x^2 \, dx + i^2 \int_0^l w_{xx}^2 \, dx - \frac{p}{E} \int_0^l w_x^2 \, dx = \text{minimum} = 0 \qquad (6.5')$$

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[1] The term $u_x^2/2$ goes out in the expansion of the radical (see equation (2.2)), and the expansion of the curvature $\frac{1}{\rho} = \frac{w_{xx}}{\left(\sqrt{1 - w_x^2} \right)^3}$ would give terms of the third order.

The expression coincides perfectly with the earlier expression (4.2_1); therefore the same differential equations and boundary conditions and especially the result $u \equiv 0$ are obtained entirely independently of whether a motion in the x–direction of the right–hand end point during the buckling is permitted or prevented. This double result (that $u = 0$, and that the buckling is independent of the condition $u(l) = 0$ or $u_x(l) = 0$) makes possible the two following "customary" interpretations of the buckling process.

The first procedure consists in considering instead of the "natural" problem, buckling under fixed load, the problem of buckling under fixed end point and at the same time (what seems almost a natural consequence of this stipulation) assuming from the start the vanishing of u also in the interior. Hence there is superimposed upon the straight position $w = 0$ a purely transverse displacement as a variation, keeping in mind the presence of the still unknown longitudinal compressive stress $-E\epsilon_0$. According to equation (2.2), as a consequence of the change in length connected with the transverse displacement, the following stretching energy is released

$$\bar{A}_1 = \int_0^l (-EF\epsilon_0)\,\bar{\epsilon}_x dx = -\frac{EF\epsilon_0}{2} \int_0^l w_x^2\, dx$$

at the same time a bending energy

$$\underline{A}_1 = \frac{EJ}{2} \int_0^l w_{xx}^2\, dx$$

must be added. This interplay between the two types of energy (and hence the two equilibrium positions) takes place when the values of $\bar{A}_1$ and $\underline{A}_1$ are numerically exactly equal; that is, when

$$A_1 = \frac{EF}{2} \left(i^2 \int_0^l w_{xx}^2\, dx - \epsilon_0 \int_0^l w_x^2\, dx \right) = 0 \qquad (6.6)$$

From the additional condition $\delta A_1 = 0$ there follows as above the sine equation for w. This procedure thus leads to the correct end result without, however, permitting a guarantee of really having found the minimal buckling load. For the "restraint" assumption $u \equiv 0$ limits the number of possible variations, and that it leads to the correct buckling load for the rod (and plate) requires at least a supplementary verification.

More important — because in a still more special way a peculiarity of the rod and plate — is another manner of thinking, which is almost universally made the basis of derivation of the buckling equations. With reference to the natural buckling process, the boundaries are considered as movable; however — and this is the characteristic mark of this method — they cannot be allowed virtual displacement u (or u, v — which, as has been observed, would subsequently become zero) but are given a displacement that is of a higher order of smallness (compared with w).

In the case of the beam it is customary to start this procedure with the assumption that no additional stretching energy is taken up during bending; that is, that the bent beam has the same length as the straight one; it follows therefrom that as a result of the bending the ends must approach each other by an amount

$$u_l = -\frac{1}{2} \int_0^l w_x^2 \, dx \qquad (6.7)$$

(which in fact is of the second order in w!), so that the external forces

do the work $-p_0 F u_l = \dfrac{p_0 F}{2} \displaystyle\int_0^l w_x^2 \, dx$. Now by formulating the equality

of inner and outer work (wherein by inner work is to be understood only the bending energy)

$$\underline{A}_i = \underline{A}_a \quad \text{or} \quad \frac{EJ}{2} \int_0^l w_{xx}^2 \, dx - \frac{EF\epsilon_0}{2} \int_0^l w_x^2 \, dx = 0$$

and assuming as above the minimal property of this expression, this procedure leads to equation (6.5') — naturally likewise without the u-term.[1]

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[1]In the assumption (6.7) there is an inconsistency: It cannot be assumed a priori that a strut that changes its length elastically below the buckling limit suddenly ceases to do so beyond it. (In reality it changes its length by quantities of higher order.) It is more logical to consider a perfectly incompressible — rigid against extension but elastic in bending — strut, for which the two hitherto independent displacements u and w are related from the beginning by the (geometrical) assumption

$$u_x + \frac{w_x^2}{2} = 0 \qquad (6.7')$$

Since the procedure of equating the stretching energy to the external work cannot be used in the case of the plate, a special auxiliary idea has been used there in order to preserve the conceptually so similar idea, that the boundaries are to move. (See reference 6.)

Without connecting the displacements u, v, w with each other numerically proceed, in this method, from the assumption that u, v are of a higher order of smallness than w; that is, consider u, v not as really independent virtual displacements but as connected with the transverse displacement w by the order-of-magnitude condition

$$u_x \approx w_x^2 \quad (\text{etc.})$$

According to equation (2.2) there is obtained for the work done by the critical stresses σ_x, σ_y, τ on the displacements u, v, w to terms of the "second order"

$$\overline{A_1} = \sigma_x s \iint \left(u_x + \frac{w_x^2}{2} \right) dx\ dy + \sigma_y s \iint \left(v_y + \frac{w_x^2}{2} \right) dx\ dy$$

$$+ \tau s \iint \left(u_y + v_x + w_x w_y \right) dx\ dy \qquad (6.8_1)$$

the bending energy is, as always, given by

$$\widetilde{A_1} = \frac{E s^3}{12(1 - \nu^2)} \iint \left[(\Delta w)^2 - 2(1 - \nu)\left(w_{xx}\, w_{yy} - w_{xy}^2 \right) \right] dx\ dy \qquad (6.8_2)$$

(s = thickness)

(Continued from p. 25)

Such a strut permits no deformation at all below the critical load; above it takes on only bending energy, which is furnished by the external work

$$\frac{1}{2} p_0^F \int_0^l w_x^2\ dx$$

Since up to the critical load no elastic deformation at all has taken place, the two laws

$$A_1 + V = \delta(A_1 + V) = 0 \quad \text{and} \quad \delta^2(A_1 + V) = \delta\left[\delta^2 (A_1 + V) \right] = 0$$

are here in content completely identical.

Both parts together must be equal to the work $\underline{A}_a$ of the external force in the sense of equation (6.3). The external work can now (and this is the essence of Reissner's idea) be expressed generally in a very simple manner, if it is remembered that the straight position is an equilibrium position and that therefore in every virtual displacement $\underline{A}_a = \underline{A}_1$. On taking the special displacement $u^* = u$, $v^* = v$, $w^* = 0$ (with the boundary displacements $u_R^* = u_R$, $v_R^* = v_R$), $\underline{A}_a = \underline{A}_1^*$; therefore

$$\underline{A}_1 = \underline{A}_1^* \equiv \iint \left[\sigma_x u_x + \sigma_y v_y + \tau \left(u_y + v_x \right) \right] dx \, dy \quad (6.8_3)$$

and now on collecting terms in (6.8), the $u-$ and $v-$terms cancel out; there results the well-known Bryan plate equation (reference 1, p. 293), exactly in the form obtained also under the assumption of purely transverse displacements and immovable boundaries.

The advantage of this method is that it offers the possibility of formulating exactly the related presentation of a solution of the buckling process by a boundary displacement. Its disadvantage is a double one: The emphasizing of the boundary displacements gives the impression that the participation of the external work is universally important in a buckling process, which, as has been seen, is not so. But besides this it is important for the entire consideration, just as for that of Bryan, that u, v are of the second order with respect to w, which must be known somehow beforehand;[1] therefore the interpretation of the external work $\sigma_x u_x \ldots$ as a contribution of the second order is not transferable to more general buckling problems. (See reference 6.)

To summarize briefly the result of this section: In considering the critical point it is customary to dispense with the correct method of writing the virtual displacements δu, δv, δw in favor of the more convenient u, v, w; thereby the connection between the customary stability criterion and the principle of virtual displacements is concealed. To be added is that the stability problem of the rod and of the plate permits a special treatment which rests upon the fact that at the critical point the tangential displacements u, v and the normal

[1]The very obvious conclusion, that can just be seen from the form (2.2) of the strain that u_x and w_x^2 must be of the same order, is not tenable; for an equation of the type (2.2) holds, for instance, also for the longitudinal fibers of a cylinder, and yet here u_x and w_x can become comparable because the tangential displacement v, which is of the same order as w, is linearly coupled with u through the shear and the transverse contraction.

displacement w are of different orders of magnitude. Since, however,
the rod or plate problem, as the analytically simplest, is at the same
time the best known, the need easily arises of transferring methods of
thinking successful in these problems to more complicated problems, which,
as was to be shown, is not possible.

THE DURCHSCHLAG — PROBLEM OF THE SLIGHTLY CURVED BEAM

In section Stable and Unstable Equilibrium, it was shown that for
the Euler strut the instability point (defined by $\delta(\delta^2 II) = 0$) coincided
with the branch point of the equilibrium. Branching problems are, however,
not the only kind of stability problem; a second class, which is just as
suited to the energy definition of stability as are the branching
problems, comprises the so-called Durchschlag problems.

In the Durchschlag problem the critical load is designated as that
load under which an (infinitesimal) displacement of the point of applica-
tion of the load is possible without an increase in the load, for which —
as in the branching problem — there are therefore two (infinitesimally
close) equilibrium positions. Above the critical point an increasing
displacement is in general accompanied by a decreasing load — the state
is unstable, the system "snaps" into[1] or falls into a stable configuration.
Prerequisite for such a phenomenon is a nonlinear relationship between
force and displacement even in the stable region.

The simplest Durchschlag problem is that of a slightly curved
beam under a transverse load. (See reference 7.)

If the ends of the initially curved beam are prevented from dis-
placing (fig. 5), then connected with the deflection caused by the
transverse force Q is a shortening of the axis of the arc, as the
result of which a horizontal force H is made to act. Because the
effect of this (very large) compressive force upon the equilibrium of
forces in an element of the beam cannot be neglected, there arise
phenomena related to the buckling process in the Euler strut, insta-
bility phenomena.

Without carrying out all the details of the calculation (presented
completely elsewhere, reference 8) the principal method of solution for
this stability problem will be briefly outlined.

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[1]In a manner similar to that in which a strut compressed beyond
the Euler limit at the least disturbance snaps or falls into the bent
position.

By a process that follows very closely that carried out in section Energy of Deformation, is obtained, with the notation of figure 5 (w now taken positive downward), for the potential energy

$$\Pi = \frac{EF}{2} \int_0^l \left[\left(u_x + \frac{w_x^2}{2} - W_x w_x \right)^2 + i^2 w_{xx}^2 \right] dx - Qf \quad (7.1)$$

From

$$\delta\Pi = EF \int_0^l \left(u_x + \frac{w_x^2}{2} - W_x w_x \right) \delta u_x dx$$

$$+ EF \int_0^l \left[\left(u_x + \frac{w_x^2}{2} - W_x w_x \right) \left(w_x - W_x \right) \delta w_x + i^2 w_{xx} \delta w_{xx} \right] dx - Q\delta f$$

there are obtained the two equilibrium conditions expressed in terms of the displacements:

$$\frac{\partial}{\partial x} \left(u_x + \frac{w_x^2}{2} - W_x w_x \right) = 0$$

or, integrated once,

$$\left(u_x + \frac{w_x^2}{2} - W_x w_x \right) = \text{constant} = - h \quad (7.2)$$

and, with the use of equation (7.2)

$$i^2 w_{xxxx} + h w_{xx} = h W_{xx} \quad (7.3)$$

also the boundary conditions

$$u(0) = u(l) = w(0) = w(l) = w_{xx}(0) = w_{xx}(l) = 0$$

$$w_{xxx}(l/2 + 0) - w_{xxx}(l/2 - 0) = Q/EJ \quad (7.4)$$

which together with the continuity requirements for

$$u, \ u_x, \ w, \ w_x, \ w_{xx}$$

at $x = l/2$ give the 12 conditions that are necessary for the evaluation of the 2×6 constants of integration in the 2 regions $x \gtrless l/2$.

The physical significance of the constant h can be recognized directly from (7.2): On the left-hand side is the stretching of the middle line of the arch; therefore, to a factor $1/EF$ h is equal to the horizontal force H, and (7.2) expresses the equilibrium condition that H does not vary with x.

Just as in the case of the Euler strut the constant of integration $h = H/EF$ of the first equation enters into the second as the coefficient of the unknown w; that is, the system (7.2) and (7.3) is nonlinear. (See equation (3.4) or (3.2)). Nevertheless, just as before the exact solution can be given in this simple case without difficulty in terms of the at first unknown

$$\alpha^2 = \frac{l^2 h}{\pi^2 i^2} = \frac{H}{H^*}, \quad H^* = \frac{EJ\pi^2}{l^2}$$

for example, for $W = f_o \sin \frac{\pi x}{l}$:

$$w = f_o \frac{\alpha^2}{\alpha^2 - 1} \sin \frac{\pi x}{l} + \frac{Ql^3}{2\pi^2 \alpha^2 EJ} \left(\frac{\sin \frac{\alpha \pi x}{l}}{\cos \frac{\alpha \pi}{2}} - \frac{\alpha \pi x}{l} \right) \qquad (7.5)$$

$$(x \leq l/2)$$

and from (7.2) by another integration taking into account the boundary conditions $u(0) = u(l) = 0$ there is obtained subsequently a transcendental equation for the dependence of α upon Q and f_o of the form:

$$\frac{H}{EF} = \frac{1}{l} \int_0^l \left(W_x w_x - \frac{w_x^2}{2} \right) dx \qquad (7.6)$$

The determination of the critical load can be carried out in two basically different ways.

The first method (see C. B. Biezeno, reference 7, pp. 21 ff) proceeds from the condition $\partial Q / \partial f = 0$, wherein the relation between Q and f is established by equations (7.6) and (7.5) for $x = l/2$

$$w(l/2) = f = f_0 \frac{\alpha^2}{\alpha^2 - 1} + \frac{Ql^3}{2\pi^2 \alpha^2 EJ} \left(\tan \frac{\pi\alpha}{2} - \frac{\pi\alpha}{2} \right) \quad (7.5')$$

A second method proceeds by way of the energy criterion (4.3). For $\delta^2\Pi$ is obtained from equation (7.1) after writing for brevity[1]

$$\left[\left(u_x + \frac{w_x^2}{2} - W_x w_x \right) + \left(W_x - w_x \right)^2 \right] = - \left[\frac{H}{EF} - \left(W_x - w_x \right)^2 \right] = -\omega(x,\alpha) \quad (7.7)$$

the expression

$$\delta^2\Pi = EF \int_0^l \left[\left(\delta u_x \right)^2 + 2 \left(w_x - W_x \right) \delta u_x \delta w_x - \omega(x,\alpha) \left(\delta w_x \right)^2 + i^2 \left(\delta w_{xx} \right)^2 \right] dx \quad (7.8)$$

The particular displacement system $\delta u, \delta w$ by which $\delta^2\Pi$ is just made equal to zero is obtained from

$$\delta(\delta^2\Pi) = 0$$

that is, from the two homogeneous differential equations

$$\delta u_{xx} + \frac{\partial}{\partial x} \left[\left(w_x - W_x \right) \delta w_x \right] = 0$$

and

$$-\frac{\partial}{\partial x} \left[\left(w_x - W_x \right) \delta u_x \right] + \frac{\partial}{\partial x} \left[i^2 \delta w_{xxx} + \omega(\alpha,x) \delta w_x \right] = 0 \quad (7.9)$$

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[1]Therein w is at first according to (7.5) a function of the two parameters a and Q. Q for example, is considered as eliminated with the help of equations (7.6) and (7.5').

with the homogeneous boundary conditions

$$\delta u(0) = \delta u(l) = \delta w(0) = \delta w(l) = \delta w_{xx}(0) = \delta w_{xx}(l) = 0$$

The desired critical α-value is the lowest eigenvalue of the equations (7.9).

APPROXIMATE DETERMINATION OF THE SNAP LOAD

Because of the great mathematical difficulties that equations (7.9) present, the second method outlined is not suitable for an exact treatment of the problem but is well suited — and therefore that procedure will be considered here — to an approximate treatment by the method of Ritz or Galerkin.

This procedure can be started at either of two points: either, make a Ritz approximation for δw in (7.9), determine the corresponding δu from the first of equations (7.9), and following Galerkin from the condition $\delta(\delta^2 \Pi) = 0$ obtain a (transcendental) equation for the determination of α; or — very much more simply, if also necessarily with a corresponding loss in accuracy — introduce at the start a Ritz approximation for w itself in place of (7.5) into the expression (7.1).

It is well to use the second method but only indicate (reference 8) the course of the calculation. If again

$$W = f_0 \sin \frac{\pi x}{l}$$

is chosen and as a Ritz expression

$$w = f_1 \sin \frac{\pi x}{l} + f_2 \sin \frac{2\pi x}{l} \tag{8.1}$$

then all boundary conditions are fulfilled, except for the one discontinuity requirement (7.4), the violation of which is, however, unimportant. Further, by satisfying exactly equation (7.2) (obtained by variation with respect to u) and calculating the horizontal force H from (7.6), the integral in (7.1) can be evaluated and Π is obtained as a function of the amplitudes f_0, f_1, f_2 or the dimensionless parameters

$$\lambda_0 = \frac{f_0}{1}, \quad \lambda_1 = \frac{f_1}{1}, \quad \lambda_2 = \frac{f_2}{1} \quad (1 = \text{radius of gyration})$$

in the simple form

$$\hat{\Pi} = \frac{4l^4}{\pi^4 i^4} \frac{\Pi}{EFl} = \lambda_1^2 + 16\lambda_2^2 + \frac{1}{2}\left(\lambda_0\lambda_1 - \frac{\lambda_1^2}{2} - 2\lambda_2^2\right)^2 - q\lambda_0\lambda_1 \quad (8.2)$$

where

$$q = \frac{1}{\lambda_0} \frac{4l^3}{\pi^4 i^3} \frac{Q}{EF} = \frac{4l}{\pi^2 f_0} \frac{Ql^2}{EJ\pi^2} = \frac{4l}{\pi^2 f_0} \frac{Q}{H^*} \quad (8.2')$$

The equilibrium equations read

$$\frac{\partial\hat{\Pi}}{\partial\lambda_1} \equiv 2\lambda_1 + (\lambda_0 - \lambda_1)\left(\lambda_0\lambda_1 - \frac{\lambda_1^2}{2} - 2\lambda_2^2\right) - q\lambda_0 = 0$$

$$(8.3)$$

$$\frac{\partial\hat{\Pi}}{\partial\lambda_2} \equiv 32\lambda_2 - 4\lambda_2\left(\lambda_0\lambda_1 - \frac{\lambda_1^2}{2} - 2\lambda_2^2\right) = 0$$

They are in the two unknowns λ_1 and λ_2 and of the third degree; nevertheless a complete discussion is possible without numerical calculation, because the second equation may be written as a product

$$\lambda_2\left(\lambda_0\lambda_1 - \frac{\lambda_1^2}{2} - 2\lambda_2^2 - 8\right) = 0$$

Therefore the cases can be distinguished

$$\lambda_2 = 0$$

$$2\lambda_1 + (\lambda_0 - \lambda_1)\left(\lambda_0\lambda_1 - \frac{\lambda_1^2}{2}\right) = q\lambda_0 \quad (8.4)$$

and $\lambda_2 \neq 0$, that is,

$$2\lambda_2^2 = \lambda_0\lambda_1 - \lambda_1^2/2 - 8$$

$$8\lambda_0 - 6\lambda_1 = q\lambda_0 \quad (8.5)$$

the discussion of which no longer requires any labor. The first system provides a symmetrical deformation, the second a superposition of a symmetrical and an antisymmetrical deformation. The corresponding horizontal compressive forces become

$$H_I = H^* \frac{\lambda_0 \lambda_1 - \lambda_1^2/2}{2} \tag{8.4'}$$

$$H_{II} = H^* \frac{\lambda_0 \lambda_1 - \lambda_1^2/2 - 2\lambda_2^2}{2} = 4H^* \tag{8.5'}$$

The critical load q_{crit} is found from the condition

$$\delta^2 \hat{\Pi} = \frac{\partial^2 \hat{\Pi}}{\partial \lambda_1^2} (\delta \lambda_1)^2 + 2 \frac{\partial^2 \hat{\Pi}}{\partial \lambda_1 \partial \lambda_2} (\delta \lambda_1 \, \delta \lambda_2) + \frac{\partial^2 \hat{\Pi}}{\partial \lambda_2^2} (\delta \lambda_2)^2 \geq 0 \tag{8.6}$$

As long as $\dfrac{\partial^2 \hat{\Pi}}{\partial \lambda_1^2}$ and the discriminant

$$\frac{\partial^2 \hat{\Pi}}{\partial \lambda_1^2} \frac{\partial^2 \hat{\Pi}}{\partial \lambda_2^2} - \left(\frac{\partial^2 \hat{\Pi}}{\partial \lambda_1 \partial \lambda_2} \right)^2$$

are greater than zero, $\delta^2 \hat{\Pi}$ as a positive-definite quadratic form in $\delta \lambda_1$ and $\delta \lambda_2$ cannot be zero for any combination of these two variables. Vanishing of the discriminant characterizes the pair of values λ_1, λ_2 for which there is exactly one combination $\delta \lambda_1, \delta \lambda_2$ for which $\delta^2 \hat{\Pi}$ becomes zero but none for which it is less than zero. The condition

$$\frac{\partial^2 \hat{\Pi}}{\partial \lambda_1^2} \frac{\partial^2 \hat{\Pi}}{\partial \lambda_2^2} - \left(\frac{\partial^2 \hat{\Pi}}{\partial \lambda_1 \partial \lambda_2} \right)^2 = 0 \tag{8.7}$$

gives, therefore, the stability limit and together with the two equations (8.3) determines the three unknowns λ_1, λ_2, and q_{crit}. In this case (8.7) reads

$$4 \left[8 - \left(\lambda_0 \lambda_1 - \frac{\lambda_1^2}{2} - 2\lambda_2^2 \right) + 4\lambda_2^2 \right] \left[2 + (\lambda_0 - \lambda_1)^2 - \left(\lambda_0 \lambda_1 - \frac{\lambda_1^2}{2} \right) \right.$$

$$\left. + 2\lambda_2^2 \right] - 16\lambda_2^2 (\lambda_0 - \lambda_1)^2 = 0$$

which condition becomes

$$\lambda_2 = 0.$$

with

$$4 \left[8 - \left(\lambda_0 \lambda_1 - \frac{\lambda_1^2}{2} \right) \right] \left[2 + (\lambda_0 - \lambda_1)^2 - \left(\lambda_0 \lambda_1 - \frac{\lambda_1^2}{2} \right) \right] = 0$$

and with

$$2\lambda_2^2 = \lambda_0 \lambda_1 - \frac{\lambda_1^2}{2} - 8$$

$$- 48 \left(\lambda_0 \lambda_1 - \lambda_1^2/2 - 8 \right) = 0$$

There are therefore two sets of values for λ_1, λ_2, and q_{crit}

$$\lambda_2 = 0, \; \lambda_1 = \lambda_0 - \frac{1}{\sqrt{3}} \sqrt{\lambda_0^2 - 4}, \quad q_{crit} = 2 + \frac{1}{\lambda_0} \left(\frac{\lambda_0^2 - 4}{3} \right)^{3/2} \qquad (8.8)$$

$$\lambda_2 = 0, \; \lambda_1 = \lambda_0 - \sqrt{\lambda_0^2 - 16}, \quad q_{crit} = 2 - \frac{6}{\lambda_0} \sqrt{\lambda_0^2 - 16} \qquad (8.9)$$

by which a critical state of the elastic system is characterized.

The physical significance of the relations (8.8), (8.9) and espe-
cially the (in this case unstable) behavior above the critical load will
not be pursued in detail (see reference 8). (See fig. 6.) There the
load Q is plotted against the deflection f_1 of the point of application,

with the initial amplitude f_o as a parameter. For $f_o/1 = \lambda_o \lesssim 2$ Q increases monotonically with f_1; instability is not possible. In the region

$$2 < \lambda_o \leq \sqrt{22} \qquad\qquad (8.8')$$

under the critical load given by (8.8) there enters definite instability, increasing deflection f_1 without increase of load. Accordingly the beam snaps — under constant load — until it finds a stable configuration[1] at λ_1'. Since $\lambda_1' > \lambda_o$ the beam is now convex downward; it also can be seen clearly that the system now must be stable with respect to an increase in load; a further deflection results in a longitudinal pull. (See equation (8.4').)

For

$$\lambda_o \geq \sqrt{22} \qquad\qquad (8.9')$$

the critical load is given by (8.9). Before the external load can assume the value (8.8), the longitudinal compression according to (8.5') reaches the value $4H^*$, that is, the second Euler load, under which the strut assumes the S-shape configuration $\lambda_2 \neq 0$. It snaps again into a stable position λ_1', this time, however, passing through an unsymmetrical deformation. At the critical load there appears a branching of the elastic equilibrium; figure 6 shows the two branches of the $Q - f_1$ curve, both of which however — and this is the noteworthy difference from the Euler problem — are unstable.

For further details see the publication referred to. Here it was just a question of presenting the chain of ideas that led to the determination of the critical loads (8.8) and (8.9), in order to show the application of the general stability criterion (4,3) to a stability problem of an entirely different kind.

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[1]That is, at first vibrates about λ_1' as a stable equilibrium position.

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TRANSLATOR'S NOTES

Equation 3.7, last part

Trans. note: It appears that this equation should be

$$u = -\epsilon^* x - \frac{\pi}{8} \frac{f^2}{l} \sin \frac{2\pi x}{l} - \frac{f^2 \pi^2}{4l^2} x$$

Equations (7.5) and (7.5')

Trans. note: It appears that the term $\dfrac{Ql^3}{2\pi^2 \alpha^2 EJ}$ should be $\dfrac{Ql^3}{2\pi^3 \alpha^3 EJ}$.

Equation (8.8)

Trans. note: It appears that $2 + \dfrac{1}{\lambda_0}\left(\dfrac{\lambda_0^2 - 4}{3}\right)^{3/2}$ should be

$$2 + \frac{2\sqrt{3}}{\lambda_0}\sqrt{\lambda_0^2 - 4}$$

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Trans. note: It appears that $\underline{A}_1 = \dfrac{EJ}{2} \displaystyle\int_0^l w_{xx}^2 dx$ should be

$$\underset{\sim}{\underline{A}}_1 = \frac{EJ}{2} \int_0^l w_{xx}^2 dx$$

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Trans. note: It appears that (8.4') should be (8.5').

Figure 1.- Strut under compression.
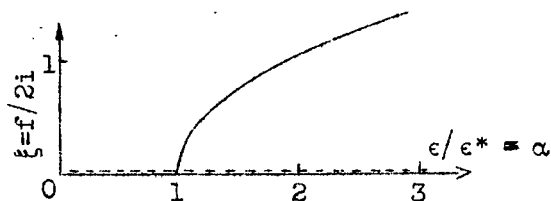
Figure 2.- Amplitude f against ϵ.
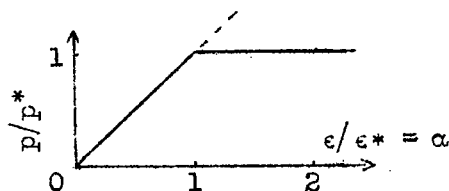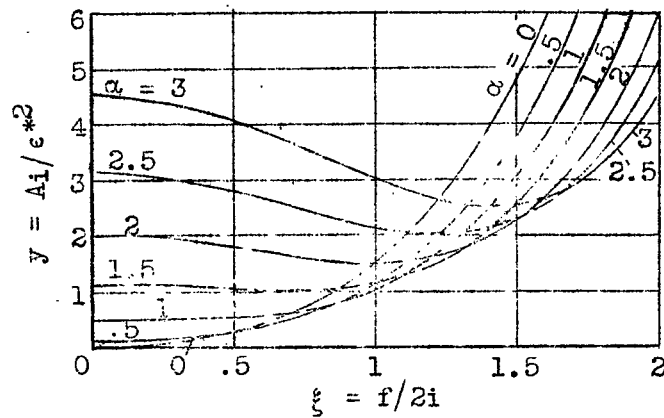

Figure 3.- Load P against ϵ.

Figure 4.- Variation of energy of deformation
            with the amplitude f = 2iξ. Edge
compression ε = αε* as a parameter.

Moment of inertia J, Section F
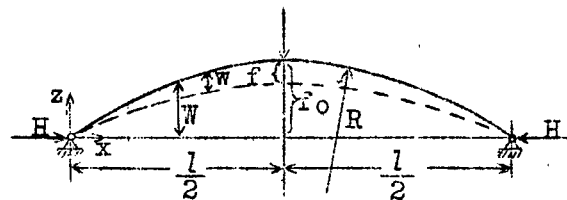Radius of gyration i, Elasticity modulus E

Figure 5.- Slightly curved beam under
              transverse load Q.
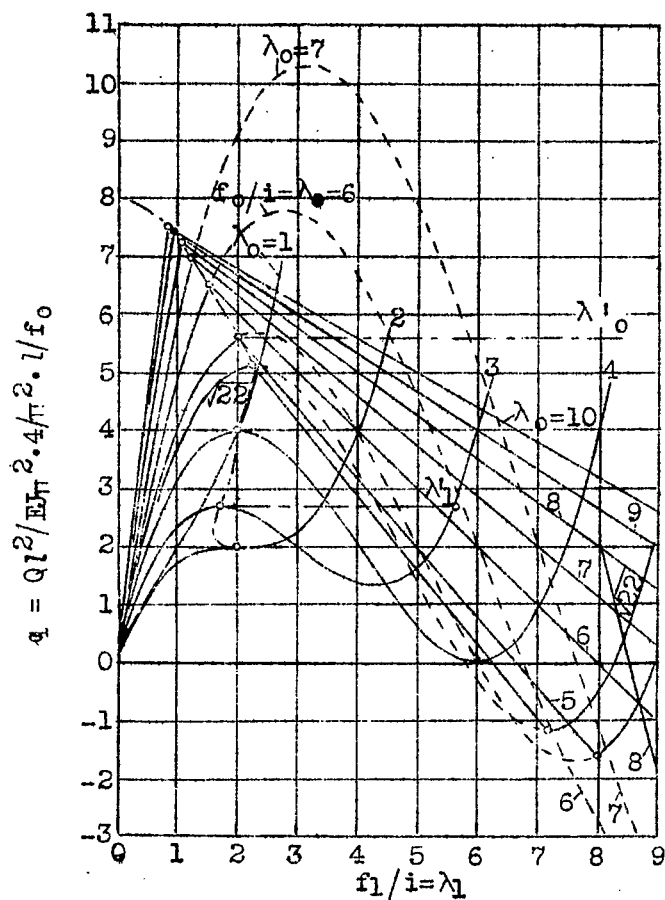
Figure 6.— Variation of the load Q with the displacement f_1 of the point of application. Parameter = initial amplitude f_0.